Mathematica Moravica
Vol. **19-1** (2015), 1–17

On the Non-Commutative Neutrix Convolution of $\tan^{-1} x$ and x^r

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Abstract. We define regular distributions $\tan_+^{-1} x$ and $\tan_-^{-1} x$ from the function $\tan^{-1} x$. We then evaluate some convolutions and neutrix convolutions of these distributions and the functions x_+^r, x_-^r and x^r.

1. Introduction

In distribution theory, the convolution of distributions is rather restricted, see for example Gel'fand and Shilov [4]. Jones [6] extended the definition of the convolution of two distributions to cover certain pairs of distributions which could not be convolved in the sense of Gel'fand and Shilov. For instance, he proved that

$$1 * \operatorname{sgn} x = x. \tag{1}$$

Fisher [3] gave a generalisation of (1) by using the Jones's definition as follow:

$$x^r * (\operatorname{sgn} x . x^r) = \frac{(r!)^2}{(2r+1)!} x^{2r+1}$$

for $r = 0, 1, 2, \ldots$. He later made some minor changes in Jones's definitions by using neutrix calculus, see [5]. He proved in [2] that

$$x_- \circledast x_+ = \frac{1}{6} \quad \text{and} \quad x_+ \circledast x_- = \frac{1}{6} x_+^3.$$

The idea of using neutrix calculus has opened up the research area in another direction for the field of distributional convolutions. In this research, it is of particular interest to evaluate convolution of distributions for a certain class of distributions.

In the following the locally summable functions x_+^r, $x_-^r, r = 0, 1, \ldots,$ $\tan_+^{-1} x$ and $\tan_-^{-1} x$ are defined by

2010 *Mathematics Subject Classification.* 46F10.
Key words and phrases. Distribution, convolution, neutrix, neutrix convolution.

- $x_+^r = \begin{cases} x^r, & x > 0, \\ 0, & x < 0, \end{cases}$
- $\tan_+^{-1} x = H(x) \tan^{-1} x$,
- $x_-^r = \begin{cases} |x|^r, & x < 0, \\ 0, & x > 0, \end{cases}$
- $\tan_-^{-1} x = H(-x) \tan^{-1} x$,

where H denotes Heaviside's function. Note that

$$x^r = x_+^r + (-1)^r x_-^r, \tag{2}$$

$$\tan^{-1} x = \tan_+^{-1} x + \tan_-^{-1} x. \tag{3}$$

If f and g are locally summable functions then the classical definition for the convolution $f * g$ of f and g is as follows:

Definition 1.1. Let f and g be functions. Then the convolution $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t) g(x-t) \mathrm{d}t \tag{4}$$

for all points x for which the integral exists.

It follows easily from the definition that if the classical convolution $f * g$ of f and g exists, then $g * f$ exists and

$$f * g = g * f. \tag{5}$$

Further, if $(f * g)'$ and $f * g'$ (or $f' * g$) exist, then

$$(f * g)' = f * g' \quad (\text{or } f' * g). \tag{6}$$

We now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$.

The classical definition of the convolution can be extended to define the convolution $f * g$ of two distributions f and g in $\mathcal{D}'$ with the following definition, see Gel'fand and Shilov [4].

Definition 1.2. Let f and g be distributions in $\mathcal{D}'$. Then the convolution $f * g$ is defined by the equation

$$\langle (f * g)(x), \varphi(x) \rangle = \langle f(y), \langle g(x), \varphi(x+y) \rangle \rangle \tag{7}$$

for arbitrary φ in $\mathcal{D}'$, provided that f and g satisfy either of the following conditions:

(a) either f or g has bounded support,
(b) the supports of f and g are bounded on the same side.

It follows that if the convolution $f * g$ exists by Definition 1.2, then (5) and (6) are satisfied.

We need the following lemmas for proving the results on the convolution.

Lemma 1.1. *If*

$$I_m = \int \tan^m x \mathrm{d}x \tag{8}$$

$m = 0, 1, 2, \dots,$ *then*

$$I_{2m} = \sum_{i=1}^{m} \frac{(-1)^{m-i}}{2i-1} \tan^{2i-1} x + (-1)^m x + constant, \tag{9}$$

$$I_{2m+1} = \sum_{i=1}^{m} \frac{(-1)^{m-i}}{2i} \tan^{2i} x + \frac{(-1)^m}{2} \ln(1 + \tan^2 x) + constant, \tag{10}$$

for $m = 0, 1, 2, \dots,$ *where the sums are empty when* $m = 0$.

Proof. It is obvious when $m = 0$ and $m = 1$ that

$$I_0 = x + \text{constant}, \tag{11}$$

$$I_1 = \int \tan x \mathrm{d}x = -\ln|\cos x| + \text{constant}$$

$$= \frac{1}{2} \ln(1 + \tan^2 x) + \text{constant}, \tag{12}$$

so that equations (8) and (9) hold when $m = 0$.

When $m \geq 2$, we have

$$I_m = \int \tan^m x \mathrm{d}x = \frac{1}{m-1} \tan^{m-1} x - I_{m-2}. \tag{13}$$

In particular, we have

$$I_{2m+2} = \frac{1}{2m+1} \tan^{2m+1} x - I_{2m}$$

and on assuming that equation (8) holds for some m, it follows that equation (8) holds for $m + 1$. Equation (8) follows by induction.

Equation(9) follows similarly. This completes the proof of the lemma. □

Lemma 1.2. *If*

$$T_k(x) = \int_0^x t^k \tan^{-1} t \mathrm{d}t, \tag{14}$$

$k = 0, 1, 2, \dots,$ *then*

(15)

$$T_{2k}(x) = \frac{1}{2k+1} \left[x^{2k+1} \tan^{-1} x - \sum_{i=1}^{k} \frac{(-1)^{k-i}}{2i} x^{2i} + \frac{(-1)^{k+1}}{2} \ln(1 + x^2) \right],$$

where the sum is empty when $k = 0$ *and*

(16)

$$T_{2k+1}(x) = \frac{1}{2k+2} \left[x^{2k+2} \tan^{-1} x - \sum_{i=0}^{k} \frac{(-1)^{k-i}}{2i+1} x^{2i+1} + (-1)^k \tan^{-1} x \right],$$

where the sum is being empty when $k = 0, 1$.

Proof. Integrating by parts by parts and then making the substitution $t = \tan u$, we have

$$\begin{aligned} T_k(x) &= \int_0^x t^k \tan^{-1} t\mathrm{d}t \\ &= \frac{1}{k+1} x^{k+1} \tan^{-1} x - \frac{1}{k+1} \int_0^{\tan^{-1} x} \tan^{k+1} u\mathrm{d}u. \end{aligned}$$

It follows that

$$T_{2k}(x) = \frac{1}{2k+1} x^{2k+1} \tan^{-1} x - \frac{1}{2k+1} \int_0^{\tan^{-1} x} \tan^{2k+1} u\mathrm{d}u, \tag{17}$$

$$T_{2k+1}(x) = \frac{1}{2k+2} x^{2k+2} \tan^{-1} x - \frac{1}{2k+2} \int_0^{\tan^{-1} x} \tan^{2k+2} u\mathrm{d}u, \tag{18}$$

for $k = 0, 1, 2. \ldots$.

It now follows from equations (10) and (17) that

$$T_{2k}(x) = \frac{1}{2k+1} \left[x^{2k+1} \tan^{-1} x - \sum_{i=1}^{k} \frac{(-1)^{k-i}}{2i} x^{2i} + \frac{(-1)^{k+1}}{2} \ln(1+x^2) \right].$$

Similarly, it follows from equations (9) and (18) that

$$T_{2k+1}(x) = \frac{1}{2k+2} \left[x^{2k+2} \tan^{-1} x - \sum_{i=0}^{k} \frac{(-1)^{k-i}}{2i+1} x^{2i+1} + (-1)^k \tan^{-1} x \right].$$

This completes the proof of the lemma. □

Theorem 1.1. *The convolution $(\tan_+^{-1} x) * x_+^r$ exists and*

$$(\tan_+^{-1} x) * x_+^r = \sum_{k=0}^{r} \binom{r}{k} (-1)^k x_+^{r-k} T_k(x), \tag{19}$$

for $r = 0, 1, 2, \ldots$.

Proof. It is obvious that $(\tan_+^{-1} x) * x_+^r = 0$ if $x < 0$. When $x > 0$, we have

$$\begin{aligned} (\tan_+^{-1} x) * x_+^r &= \int_0^x \tan^{-1} t(x-t)^r \mathrm{d}t \\ &= \sum_{k=0}^{r} \binom{r}{k} (-1)^k x^{r-k} \int_0^x t^k \tan^{-1} t\mathrm{d}t \\ &= \sum_{k=0}^{r} \binom{r}{k} (-1)^k x^{r-k} T_k(x), \end{aligned}$$

proving equation (19). □

Corollary 1.1. *The convolution* $(\tan_-^{-1} x) * x_-^r$ *exists and*

$$(\tan_-^{-1} x) * x_-^r = -\sum_{k=0}^{r} \binom{r}{k} x_-^{r-k} T_k(x), \tag{20}$$

for $r = 0, 1, 2, \ldots$.

Proof. Equation (20) follows on replacing x by $-x$ in equation (19) on noting that $T_k(-x) = (-1)^k T_k(x)$. □

Corollary 1.2. *The convolution* $(1+x^2)_+^{-1} * x_+^r$ *and* $(1+x^2)_-^{-1} * x_-^r$ *exist and*

$$(1+x^2)_+^{-1} * x_+^r = r\sum_{k=0}^{r-1} \binom{r-1}{k} (-1)^k x_+^{r-k-1} T_k(x), \tag{21}$$

$$(1+x^2)_-^{-1} * x_-^r = r\sum_{k=0}^{r-1} \binom{r-1}{k} x_-^{r-k-1} T_k(x), \tag{22}$$

for $r = 0, 1, 2, \ldots$, *where the sums are empty when* $r = 0$.

Proof. Equation (21) follows from equations (6) and (19).

Equation (22) follows from equations (6) and (20). □

2. The Neutrix Convolution

The definition of the convolution of distributions is rather restrictive and so the non-commutative neutrix convolution of distributions was introduced in [1]. In order to define the neutrix convolution product we first of all let τ be a function in $\mathcal{D}$ satisfying the following properties:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \le \tau(x) \le 1$,
(iii) $\tau(x) = 1$ for $|x| \le \frac{1}{2}$,
(iv) $\tau(x) = 0$ for $|x| \ge 1$.

The function τ_n is then defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \le n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

for $n = 1, 2, \ldots$.

The following definition was given in [1].

Definition 2.1. Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ for $n = 1, 2, \ldots$. Then the *neutrix convolution* $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, provided that the limit h exists in the sense

$$\operatorname*{N-lim}_{n\to\infty} \langle f_n * g, \varphi\rangle = \langle h, \varphi\rangle$$

for all φ in $\mathcal{D}$, where N is the neutrix, see van der Corput [5], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range N'', the real numbers, with negligible functions being finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n \quad (\lambda > 0, r = 1, 2, \ldots)$$

and all functions which converge to zero in the usual sense as n tends to infinity.
In particular, if

$$\lim_{n\to\infty} \langle f_n * g, \varphi \rangle = \langle h, \varphi \rangle$$

for all φ in $\mathcal{D}$, we say that the *convolution* $f * g$ exists and equals h.

Note that in this definition the convolution $f_n * g$ is as defined in Gel'fand and Shilov's sense since the distribution f_n having compact support. Note also that because of the lack of symmetry in the definition of $f \circledast g$, the neutrix convolution is in general non-commutative.

The following theorem was proved in [1], showing that the neutrix convolution is a generalization of the convolution.

Theorem 2.1. *Let f and g be distributions in $\mathcal{D}'$ satisfying either condition (a) or condition (b) of Gel'fand and Shilov's definition. Then the neutrix convolution $f \circledast g$ exists and*

$$f \circledast g = f * g.$$

The next theorem was also proved in [1].

Theorem 2.2. *Let f and g be distributions in $\mathcal{D}'$ and suppose that $f \circledast g$ exists, then the neutrix convolution $f \circledast g'$ exists and*

$$(f \circledast g)' = f \circledast g'.$$

Note however that $(f \circledast g)'$ is not necessarily equal to $f' \circledast g$, but we do have the following theorem, which was proved in [2].

Theorem 2.3. *Let f and g be distributions in $\mathcal{D}'$ and suppose that $f \circledast g$ exists. If $\mathrm{N}\!-\!\lim_{n\to\infty}\langle (f\tau_n') * g, \varphi \rangle$ exists and equals $\langle h, \varphi \rangle$ for all φ in $\mathcal{D}$, then the neutrix convolution $f' \circledast g$ exists and*

$$(f \circledast g)' = f' \circledast g + h.$$

We also need the following lemmas for proving results on the non-commutative neutrix products.

Lemma 2.1.

$$\mathrm{N}\!-\!\lim_{n\to\infty}(n^r \tan^{-1} n) = \begin{cases} \dfrac{\pi}{2}, & r = 0, \\ \dfrac{(-1)^{k+1}}{2k+1}, & r = 2k+1, \\ 0, & r = 2k+2, \end{cases} \tag{23}$$

$r = 0, 1, 2, \ldots$.

Proof. Note that when n tends to infinity, we would have

$$\tan^{-1} n = \frac{\pi}{2} - \frac{1}{n} + \frac{1}{3n^3} + \cdots + \frac{(-1)^{k+1}}{(2k+1)n^{2k+1}} + O(n^{-2k-3}).$$

Equation (23) then follows. □

Lemma 2.2. *The neutrix limit of $T_{2k}(n)$ and $T_{2k+1}(n)$ exist and*

$$\operatorname*{N-lim}_{n\to\infty} T_{2k}(n) = \frac{(-1)^{k+1}}{(2k+1)^2}, \tag{24}$$

$$\operatorname*{N-lim}_{n\to\infty} T_{2k+1}(n) = \frac{(-1)^{k}\pi}{4(k+1)}, \tag{25}$$

for $k = 0, 1, 2, \ldots$.

Proof. Equation (24) follows from equations (15) and (23).
Equation (25) follows from equations (16) and (23). □

We now prove the following theorems.

Theorem 2.4. *The neutrix convolutions $(\tan_+^{-1} x) \circledast x^{2r+1}$ and $(\tan_+^{-1} x) \circledast x^{2r}$ exist and*

$$(\tan_+^{-1} x) \circledast x^{2r+1} = \sum_{k=0}^{r} \binom{2r+1}{2k} \frac{(-1)^{k+1}}{(2k+1)^2} x^{2r-2k+1} + \sum_{k=0}^{r} \binom{2r+1}{2k+1} \frac{(-1)^{k+1}\pi}{4(k+1)} x^{2r-2k}, \tag{26}$$

$$(\tan_+^{-1} x) \circledast x^{2r} = \sum_{k=0}^{r} \binom{2r}{2k} \frac{(-1)^{k+1}}{(2k+1)^2} x^{2r-2k} + \sum_{k=1}^{r} \binom{2r}{2k-1} \frac{(-1)^{k}\pi}{4k} x^{2r-2k+1}, \tag{27}$$

for $r = 0, 1, 2, \ldots$.

Proof. Put $(\tan_+^{-1} x)_n = (\tan_+^{-1} x)\tau_n(x)$. Then the convolution $(\tan_+^{-1} x)_n * x^{2r+1}$ exists and
(28)

$$\begin{aligned}(\tan_+^{-1} x)_n * x^{2r+1} &= \int_0^n \tan^{-1} t(x-t)^{2r+1} dt + \int_n^{n+n^{-n}} \tan^{-1} t(x-t)^{2r+1}\tau_n(t) dt \\ &= J_1 + J_2.\end{aligned}$$

It is easily seen that

$$\begin{aligned} J_1 &= \sum_{k=0}^{2r+1} \binom{2r+1}{k} (-1)^k x^{2r-k+1} \int_0^n t^k \tan^{-1} t dt \\ &= \sum_{k=0}^{2r+1} \binom{2r+1}{k} (-1)^k x^{2r-k+1} T_k(n) \\ &= \sum_{k=0}^{r} \binom{2r+1}{2k} x^{2r-2k+1} T_{2k}(n) - \sum_{k=0}^{r} \binom{2r+1}{2k+1} x^{2r-2k} T_{2k+1}(n). \end{aligned} \tag{29}$$

It then follows from equations (24), (25) and (29) that

$$\begin{aligned} \operatorname*{N-lim}_{n\to\infty} J_1 = \sum_{k=0}^{r} \binom{2r+1}{2k} \frac{(-1)^{k+1}}{(2k+1)^2} x^{2r-2k+1} \\ + \sum_{k=0}^{r} \binom{2r+1}{2k+1} \frac{(-1)^{k+1}\pi}{4(k+1)} x^{2r-2k}, \end{aligned} \tag{30}$$

for $r = 0, 1, 2, \ldots$.

Further, it is easily seen that

$$\lim_{n\to\infty} J_2 = 0. \tag{31}$$

Equation (26) then follows from equations (28), (30) and (31).

Equation (27) follows from equation (26) on using Theorem 2.2. □

Corollary 2.1. *The neutrix convolutions* $(\tan_-^{-1} x) \circledast x^{2r+1}$, $(\tan_-^{-1} x) \circledast x^{2r}$, $(\tan^{-1} x) \circledast x^{2r+1}$ *and* $(\tan^{-1} x) \circledast x^{2r}$ *exist and*

$$\begin{aligned} (\tan_-^{-1} x) \circledast x^{2r+1} = \sum_{k=0}^{r} \binom{2r+1}{2k} \frac{(-1)^k}{(2k+1)^2} x^{2r-2k+1} \\ + \sum_{k=0}^{r} \binom{2r+1}{2k+1} \frac{(-1)^{k+1}\pi}{4(k+1)} x^{2r-2k}, \end{aligned} \tag{32}$$

$$\begin{aligned} (\tan_-^{-1} x) \circledast x^{2r} = \sum_{k=0}^{r} \binom{2r}{2k} \frac{(-1)^k}{(2k+1)^2} x^{2r-2k} \\ + \sum_{k=1}^{r} \binom{2r}{2k-1} \frac{(-1)^k\pi}{4k} x^{2r-2k+1}, \end{aligned} \tag{33}$$

$$(\tan^{-1} x) \circledast x^{2r+1} = \sum_{k=0}^{r} \binom{2r+1}{2k+1} \frac{(-1)^{k+1}\pi}{2(k+1)} x^{2r-2k}, \tag{34}$$

$$(\tan^{-1} x) \circledast x^{2r} = \sum_{k=1}^{r} \binom{2r}{2k-1} \frac{(-1)^k\pi}{2k} x^{2r-2k+1}, \tag{35}$$

for $r = 0, 1, 2, \ldots$.

Proof. Equations (32) and (33) follow on replacing x by $-x$ in equations (26) and (27), respectively.

Equation (34) follows from equations (3), (26) and (32).

Equation (35) follows from equations (3), (27) and (33). □

Corollary 2.2. *The neutrix convolutions* $(\tan_+^{-1} x) \circledast x_-^{2r+1}$, $(\tan_+^{-1} x) \circledast x_-^{2r}$, $(\tan_-^{-1} x) \circledast x_+^{2r+1}$ *and* $(\tan_-^{-1} x) \circledast x_+^{2r}$ *exist and*

$$(\tan_+^{-1} x) \circledast x_-^{2r+1} = \sum_{k=0}^{r} \binom{2r+1}{2k} \frac{(-1)^k}{(2k+1)^2} x^{2r-2k+1}$$

$$+ \sum_{k=0}^{r} \binom{2r+1}{2k+1} \frac{(-1)^k \pi}{4(k+1)} x^{2r-2k} + \sum_{k=0}^{2r+1} \binom{2r+1}{k} (-1)^k x_+^{2r-k+1} T_k(x), \tag{36}$$

$$(\tan_+^{-1} x) \circledast x_-^{2r} = \sum_{k=0}^{r} \binom{2r}{2k} \frac{(-1)^{k+1}}{(2k+1)^2} x^{2r-2k}$$

$$+ \sum_{k=1}^{r} \binom{2r}{2k-1} \frac{(-1)^k \pi}{4k} x^{2r-2k+1} - \sum_{k=0}^{2r} \binom{2r}{k} (-1)^k x_+^{2r-k} T_k(x), \tag{37}$$

$$(\tan_-^{-1} x) \circledast x_+^{2r+1} = \sum_{k=0}^{r} \binom{2r+1}{2k} \frac{(-1)^k}{(2k+1)^2} x^{2r-2k+1}$$

$$+ \sum_{k=0}^{r} \binom{2r+1}{2k+1} \frac{(-1)^{k+1} \pi}{4(k+1)} x^{2r-2k} - \sum_{k=0}^{2r+1} \binom{2r+1}{k} x_-^{2r-k+1} T_k(x), \tag{38}$$

$$(\tan_-^{-1} x) \circledast x_+^{2r} = \sum_{k=0}^{r} \binom{2r}{2k} \frac{(-1)^k}{(2k+1)^2} x^{2r-2k}$$

$$+ \sum_{k=1}^{r} \binom{2r}{2k-1} \frac{(-1)^k \pi}{4k} x^{2r-2k+1} + \sum_{k=0}^{2r} \binom{2r}{k} x_-^{2r-k} T_k(x), \tag{39}$$

for $r = 0, 1, 2, \ldots$.

Proof. We have from equation (2) that

$$(\tan_+^{-1} x) \circledast x^r = [(\tan_+^{-1} x) * x_+^r] + (-1)^r [(\tan_+^{-1} x) \circledast x_-^r], \tag{40}$$

for $r = 0, 1, 2, \ldots$.

Equation (36) follows from equations (19), (26) and (40).

Equation (37) follows from equations (19), (27) and (40).

Equations (38) and (39) follow on replacing x by $-x$ in equations (36) and (37), respectively, on noting that $T_k(-x) = (-1)^k T_k(x)$. □

Theorem 2.5. *The neutrix convolutions* $(1+x^2)_+^{-1} \circledast x^{2r+1}$ *and* $(1+x^2)_+^{-1} \circledast x^{2r}$ *exist and*

$$(1+x^2)_+^{-1} \circledast x^{2r+1} = \sum_{k=0}^{r} \binom{2r+1}{2k} \frac{(-1)^{k+1}(2r-2k+1)}{(2k+1)^2} x^{2r-2k} + \sum_{k=0}^{r} \binom{2r+1}{2k+1} \frac{(-1)^{k+1}(r-k)\pi}{2(k+1)} x^{2r-2k-1} + \sum_{k=0}^{r} \binom{2r+1}{2k+1} \frac{(-1)^k}{2k+1} x^{2r-2k} + \frac{\pi}{2} x^{2r+1}, \tag{41}$$

$$(1+x^2)_+^{-1} \circledast x^{2r} = \sum_{k=0}^{r} \binom{2r}{2k} \frac{(-1)^{k+1}2(r-k)}{(2k+1)^2} x^{2r-2k-1} + \sum_{k=0}^{r-1} \binom{2r}{2k+1} \frac{(-1)^{k+1}(2r-2k-1)\pi}{4(k+1)} x^{2(r-k-1)} + \sum_{k=0}^{r} \binom{2r}{2k+1} \frac{(-1)^k}{2k+1} x^{2r-2k-1} + \frac{(2r+1)\pi}{2} x^{2r}, \tag{42}$$

for $r = 0, 1, 2 \ldots$.

Proof. Put $[\tan_+^{-1} x]_n = (\tan_+^{-1} x)\tau_n'(x)$. Then the convolution $[\tan_+^{-1} x]_n * x^{2r+1}$ exists and

$$[\tan_+^{-1} x]_n * x^{2r+1} = \int_n^{n+n^{-n}} \tan^{-1} t (x-t)^{2r+1} \tau_n'(t) \mathrm{d}t.$$

If now φ is an arbitrary function in $\mathcal{D}$ whose support containing in the interval $[a,b]$, we have

$$\langle [\tan^{-1} x_+]_n * x^{2r+1}, \varphi(x) \rangle = \int_a^b \int_n^{n+n^{-n}} \tan^{-1} t (x-t)^{2r+1} \tau_n'(t) \varphi(x) \mathrm{d}t \mathrm{d}x$$

$$= \int_a^b \int_n^{n+n^{-n}} \big[(2r+1)(x-t)^{2r} \tan^{-1} t - (1+t^2)^{-1}(x-t)^{2r-1}\big] \tau_n(t) \varphi(x) \mathrm{d}t \mathrm{d}x - \tan^{-1} n \int_a^b (x-n)^{2r+1} \varphi(x) \mathrm{d}x, \tag{43}$$

where

$$\tan^{-1} n \int_a^b (x-n)^{2r+1} \varphi(x) \mathrm{d}x = \sum_{k=0}^{2r+1} \binom{2r+1}{k} (-1)^k n^k \tan^{-1} n \langle x^{2r-k+1}, \varphi(x) \rangle$$

$$= \sum_{k=0}^{r} \binom{2r+1}{2k} n^{2k} \tan^{-1} n \langle x^{2r-2k+1}, \varphi(x) \rangle$$

$$-\sum_{k=0}^{r}\binom{2r+1}{2k+1}n^{2k+1}\tan^{-1}n\langle x^{2r-2k},\varphi(x)\rangle. \tag{44}$$

Applying the neutrix limit in equation (44) and using equation (23), give

$$\begin{aligned}&\operatorname*{N-lim}_{n\to\infty}\tan^{-1}n\int_a^b(x-n)^{2r+1}\varphi(x)\mathrm{d}x\\&=\left\langle\frac{\pi}{2}x^{2r+1}-\sum_{k=0}^{r}\binom{2r+1}{2k+1}\frac{(-1)^{k+1}}{2k+1}x^{2r-2k},\varphi(x)\right\rangle.\end{aligned} \tag{45}$$

It follows from equations (43), (44) and (45) that

$$\begin{aligned}&\operatorname*{N-lim}_{n\to\infty}\langle[\tan_+^{-1}x]_n*x^{2r+1},\varphi(x)\rangle\\&=\left\langle\sum_{k=0}^{r}\binom{2r+1}{2k+1}\frac{(-1)^{k+1}}{2k+1}x^{2r-2k}-\frac{\pi}{2}x^{2r+1},\varphi(x)\right\rangle.\end{aligned} \tag{46}$$

We have from Theorem 2.2 that

$$[(\tan_+^{-1}x)\circledast x^{2r+1}]'=[(1+x^2)_+^{-1}\circledast x^{2r+1}]+h(x), \tag{47}$$

where

$$h(x)=\sum_{k=0}^{r}\binom{2r+1}{2k+1}\frac{(-1)^{k+1}}{2k+1}x^{2r-2k}-\frac{\pi}{2}x^{2r+1}. \tag{48}$$

Equation (41) then follows from equations (26),(47) and (48).

Equation (42) follows from equation (41) on using Theorem 2.2. □

Corollary 2.3. *The neutrix convolutions* $(1+x^2)_-^{-1}\circledast x^r$ *and* $(1+x^2)^{-1}\circledast x^r$ *exist and*

$$\begin{aligned}(1+x^2)_-^{-1}\circledast x^{2r+1}&=\sum_{k=0}^{r}\binom{2r+1}{2k}\frac{(-1)^k(2r-2k+1)}{(2k+1)^2}x^{2r-2k}\\&+\sum_{k=0}^{r}\binom{2r+1}{2k+1}\frac{(-1)^{k+1}(r-k)\pi}{2(k+1)}x^{2r-2k-1}\\&+\sum_{k=0}^{r}\binom{2r+1}{2k+1}\frac{(-1)^{k+1}}{2k+1}x^{2r-2k}+\frac{\pi}{2}x^{2r+1},\end{aligned} \tag{49}$$

$$\begin{aligned}(1+x^2)_-^{-1}\circledast x^{2r}&=\sum_{k=0}^{r}\binom{2r}{2k}\frac{(-1)^k2(r-k)}{(2k+1)^2}x^{2r-2k-1}\\&+\sum_{k=0}^{r}\binom{2r}{2k+1}\frac{(-1)^{k+1}(2r-2k-1)\pi}{4(k+1)}x^{2(r-k-1)}\\&+\sum_{k=0}^{r-1}\binom{2r}{2k+1}\frac{(-1)^{k+1}}{2k+1}x^{2r-2k-1}+\frac{(2r+1)\pi}{2}x^{2r},\end{aligned} \tag{50}$$

(51)

$$(1+x^2)^{-1} \circledast x^{2r+1} = \sum_{k=0}^{r} \binom{2r+1}{2k+1} \frac{(-1)^{k+1}(r-k)\pi}{k+1} x^{2r-2k-1} + \pi x^{2r+1},$$

$$(1+x^2)^{-1} \circledast x^{2r} = \sum_{k=0}^{r-1} \binom{2r}{2k+1} \frac{(-1)^{k+1}(2r-2k-1)\pi}{2(k+1)} x^{2(r-k-1)}$$

(52) $$+ (2r+1)\pi x^{2r},$$

for $r = 0, 1, 2, \ldots$.

Proof. Equations (49) and (50) follow on replacing x by $-x$ in equations (41) and (42), respectively.

Equation (51) follows from equations (41) and (49) on noting that

$$(1+x^2)^{-1} \circledast x^{2r+1} = (1+x^2)_+^{-1} \circledast x^{2r+1} + (1+x^2)_-^{-1} \circledast x^{2r+1}.$$

Equation (52) follows from equations (42) and (50) on noting that

$$(1+x^2)^{-1} \circledast x^{2r} = (1+x^2)_+^{-1} \circledast x^{2r} + (1+x^2)_-^{-1} \circledast x^{2r}.$$

□

Corollary 2.4. *The neutrix convolutions* $(1+x^2)_+^{-1} \circledast x_-^r$ *and* $(1+x^2)_-^{-1} \circledast x_+^r$ *exist and*

$$(1+x^2)_+^{-1} \circledast x_-^{2r+1} = \sum_{k=0}^{r} \binom{2r+1}{2k} \frac{(-1)^k(2r-2k+1)}{(2k+1)^2} x^{2r-2k}$$

$$+ \sum_{k=0}^{r} \binom{2r+1}{2k+1} \left[\frac{(-1)^k(r-k)\pi}{2(k+1)} x^{2r-2k-1} + \frac{(-1)^{k+1}}{2k+1} x^{2r-2k} \right]$$

(53)

$$- \frac{\pi}{2} x^{2r+1} + (2r+1) \sum_{k=0}^{2r} \binom{2r}{k} (-1)^k x_+^{2r-k} T_k(x),$$

$$(1+x^2)_+^{-1} \circledast x_-^{2r} = \sum_{k=0}^{r} \binom{2r}{2k} \frac{(-1)^{k+1} 2(r-k)}{(2k+1)^2} x^{2r-2k-1}$$

$$+ \sum_{k=0}^{r-1} \binom{2r}{2k+1} \left[\frac{(-1)^{k+1}(2r-2k-1)\pi}{4(k+1)} x^{2(r-k-1)} + \frac{(-1)^k}{2k+1} x^{2r-2k-1} \right]$$

(54)

$$+ \frac{(2r+1)\pi}{2} x^{2r} - 2r \sum_{k=0}^{2r-1} \binom{2r-1}{k} (-1)^k x_+^{2r-k-1} T_k(x),$$

$$(1+x^2)_-^{-1} \circledast x_+^{2r+1} = \sum_{k=0}^{r} \binom{2r+1}{2k} \frac{(-1)^k(2r-2k+1)}{(2k+1)^2} x^{2r-2k}$$

$$+\sum_{k=0}^{r}\binom{2r+1}{2k+1}\left[\frac{(-1)^{k+1}(r-k)\pi}{2(k+1)}x^{2r-2k-1}+\frac{(-1)^{k+1}}{2k+1}x^{2r-2k}\right]$$

(55)

$$+\frac{\pi}{2}x^{2r+1}+(2r+1)\sum_{k=0}^{2r}\binom{2r}{k}x_-^{2r-k}T_k(x),$$

$$(1+x^2)_-^{-1}\circledast x_+^{2r}=\sum_{k=0}^{r}\binom{2r}{2k}\frac{(-1)^k2(r-k)}{(2k+1)^2}x^{2r-2k-1}$$

$$+\sum_{k=0}^{r-1}\binom{2r}{2k+1}\left[\frac{(-1)^{k+1}(2r-2k-1)\pi}{4(k+1)}x^{2(r-k-1)}+\frac{(-1)^{k+1}}{2k+1}x^{2r-2k-1}\right]$$

(56)

$$+\frac{(2r+1)\pi}{2}x^{2r}-2r\sum_{k=0}^{2r-1}\binom{2r-1}{k}x_-^{2r-k-1}T_k(x),$$

for $r=0,1,2,\ldots$.

Proof. We have

(57) $$(1+x^2)_+^{-1}\circledast x^{2r+1}=(1+x^2)_+^{-1}\circledast x_+^{2r+1}-(1+x^2)_+^{-1}\circledast x_-^{2r+1}.$$

Equation (53) then follows from equations (21), (41) and (57).

Equation (54) follows from equation (53) on using Theorem 2.2.

Equations (55) and (56) follow on replacing x by $-x$ in equations (53) and (54), respectively, on noting that $T_k(-x)=(-1)^kT_k(x)$. □

Lemma 2.3.

$$\mathrm{N-}\lim_{n\to\infty}[(x+n)^r\tan^{-1}(x+n)]=\begin{cases}\frac{\pi}{2}, & r=0,\\ \frac{\pi}{2}x^{2k+1}+\sum_{i=0}^{k}\frac{(-1)^{i+1}}{2i+1}x^{2(k-i)}, & r=2k+1,\\ \frac{\pi}{2}x^{2k+2}+\sum_{i=0}^{k}\frac{(-1)^{i+1}}{2i+1}x^{2(k-i)+1}, & r=2k+2,\end{cases}$$

$r=0,1,2,\ldots$.

Proof. Note that if x is fixed and n tends to infinity, we would have

$$\tan^{-1}(x+n)=\frac{\pi}{2}-\frac{1}{x+n}+\frac{1}{3(x+n)^3}+\cdots+\frac{(-1)^{k+1}}{(2k+1)(x+n)^{2k+1}}+O(n^{-2k-3}).$$

The proof of the lemma then follows. □

Lemma 2.4. *The neutrix limits of* $T_{2k}(x+n)$ *and* $T_{2k+1}(x+n)$ *exist and*

$$\mathrm{N-}\lim_{n\to\infty}T_{2k}(x+n)=\frac{\pi x^{2k+1}}{2(2k+1)}+\frac{(-1)^{k+1}}{(2k+1)^2}+\sum_{i=1}^{k}\frac{(-1)^{k-i+1}x^{2i}}{2i(2k-2i+1)}$$
$$=G_k(x),$$

for $k = 0, 1, 2, \ldots$, *where the sum is empty when* $k = 0$ *and*

$$\begin{aligned}\operatorname*{N-lim}_{n\to\infty} T_{2k+1}(x+n) &= \frac{\pi x^{2k+2}}{4(k+1)} + \frac{\pi(-1)^k}{4(k+1)} + \sum_{i=0}^{k} \frac{(-1)^{k-i+1}x^{2i+1}}{(2i+1)(2k-2i+1)} \\ &= F_k(x),\end{aligned}$$

for $k = 0, 1, 2, \ldots$.

Proof. Equation (58) follows from equations (15) and (2.3).
Equation (58) follows from equations (16) and (2.3). □

Theorem 2.6. *The neutrix convolutions* $x^{2r+1} \circledast \tan_+^{-1} x$ *and* $x^{2r} \circledast \tan_+^{-1} x$ *exist and*

(58)
$$x^{2r+1} \circledast \tan_+^{-1} x = \sum_{k=0}^{r} \binom{2r+1}{2k} x^{2r-2k+1} G_k(x) - \sum_{k=0}^{r} \binom{2r+1}{2k+1} x^{2r-2k} F_k(x),$$

(59)
$$x^{2r} \circledast \tan_+^{-1} x = \sum_{k=0}^{r} \binom{2r}{2k} x^{2r-2k} G_k(x) - \sum_{k=0}^{r-1} \binom{2r}{2k+1} x^{2r-2k-1} F_k(x),$$

for $r = 0, 1, 2, \ldots$.

Proof. We put $(x^{2r+1})_n = x^{2r+1}\tau_n(x)$ for $r = 0, 1, 2, \ldots$. Then the convolution $(x^{2r+1})_n * \tan_+^{-1} x$ exists and

$$\begin{aligned}(x^{2r+1})_n * \tan_+^{-1} x &= \int_0^{x+n} \tan^{-1} t (x-t)^{2r+1} \mathrm{d}t \\ &+ \int_{x+n}^{x+n+n^{-n}} \tan^{-1} t \tau_n(x-t)(x-t)^{2r+1} \mathrm{d}t \qquad (60)\end{aligned}$$

where

$$\begin{aligned}\int_0^{x+n} \tan^{-1} t (x-t)^{2r+1} \mathrm{d}t &= \sum_{k=0}^{2r+1} \binom{2r+1}{k} (-1)^k x^{2r-k+1} \int_0^{x+n} t^k \tan^{-1} t \mathrm{d}t \\ &= \sum_{k=0}^{2r+1} \binom{2r+1}{k} (-1)^k x^{2r-k+1} T_k(x+n) \\ &= \sum_{k=0}^{r} \binom{2r+1}{2k} x^{2r-2k+1} T_{2k}(x+n) \\ &\quad - \sum_{k=0}^{r} \binom{2r+1}{2k+1} x^{2r-2k} T_{2k+1}(x+n). \qquad (61)\end{aligned}$$

It follows from equations (58), (58) and (61) that

$$\operatorname*{N-lim}_{n\to\infty}\int_0^{x+n}\tan^{-1}t(x-t)^{2r+1}\mathrm{d}t=\sum_{k=0}^{r}\binom{2r+1}{2k}x^{2r-2k+1}G_k(x)$$

$$-\sum_{k=0}^{r}\binom{2r+1}{2k+1}x^{2r-2k}F_k(x). \tag{62}$$

Further, it is easily seen that

$$\operatorname*{N-lim}_{n\to\infty}\int_{x+n}^{x+n+n^{-n}}\tan^{-1}t\tau_n(x-t)(x-t)^{2r+1}\mathrm{d}t=0. \tag{63}$$

Equation (58) then follows from equations (60), (62) and (63).

The proof of equation (59) is similar to the proof of equation (58). □

Corollary 2.5. *The neutrix convolutions* $x^{2r+1}\circledast\tan_-^{-1}x$, $x^{2r}\circledast\tan_-^{-1}x$, $x^{2r+1}\circledast\tan^{-1}x$ *and* $x^{2r}\circledast\tan^{-1}x$ *exist and*

$$x^{2r+1}\circledast\tan_-^{-1}x=-\sum_{k=0}^{r}\binom{2r+1}{2k}x^{2r-2k+1}G_k(-x)$$

$$-\sum_{k=0}^{r}\binom{2r+1}{2k+1}x^{2r-2k}F_k(-x), \tag{64}$$

$$x^{2r}\circledast\tan_-^{-1}x=-\sum_{k=0}^{r}\binom{2r}{2k}x^{2r-2k}G_k(-x)$$

$$-\sum_{k=0}^{r-1}\binom{2r}{2k+1}x^{2r-2k-1}F_k(-x), \tag{65}$$

$$x^{2r+1}\circledast\tan^{-1}x=\frac{4^{r+1}-1}{2r+2}\pi x^{2r+2}+\sum_{k=0}^{r}\binom{2r+1}{2k+1}\frac{\pi(-1)^k x^{2r-2k}}{2k+2}, \tag{66}$$

$$x^{2r}\circledast\tan^{-1}x=\frac{2^{2r+1}-1}{2r+1}\pi x^{2r+1}+\sum_{k=0}^{r-1}\binom{2r}{2k+1}\frac{\pi(-1)^k x^{2r-2k-1}}{2k+2}, \tag{67}$$

for $r=0,1,2,\ldots$.

Proof. Equations (64) and (65) follow on replacing x by $-x$ in equations (58) and (59), respectively.

Further, we have

$$x^r\circledast\tan^{-1}x=x^r\circledast\tan_+^{-1}x+x^r\circledast\tan_-^{-1}x, \tag{68}$$

for $r=0,1,2,\ldots$ and

$$G_k(-x)=G_k(x)-\frac{\pi}{2k+1}x^{2k+1}, \tag{69}$$

$$F_k(-x) = -F_k(x) + \frac{\pi}{2k+2}x^{2k+2} + \frac{\pi(-1)^k}{2k+2}, \tag{70}$$

$k = 0, 1, 2, \ldots$.

Equation (66) follows from equations (58), (64), (68), (69) and (70).

Equation (67) follows from equations (59), (65), (68), (69) and (70). □

Corollary 2.6. *The neutrix convolutions* $x_-^{2r+1} \circledast \tan_+^{-1} x$, $x_-^{2r} \circledast \tan_+^{-1} x$, $x_+^{2r+1} \circledast \tan_-^{-1} x$ *and* $x_+^{2r} \circledast \tan_-^{-1} x$ *exist and*

$$\begin{aligned} x_-^{2r+1} \circledast \tan_+^{-1} x = &-\sum_{k=0}^{r} \binom{2r+1}{2k} x^{2r-2k+1} G_k(x) + \sum_{k=0}^{r} \binom{2r+1}{2k+1} x^{2r-2k} F_k(x) \\ &+ \sum_{k=0}^{2r+1} \binom{2r+1}{k} (-1)^k x_+^{2r-k+1} T_k(x), \end{aligned} \tag{71}$$

$$\begin{aligned} x_-^{2r} \circledast \tan_+^{-1} x = &\sum_{k=0}^{r} \binom{2r}{2k} x^{2r-2k} G_k(x) - \sum_{k=0}^{r-1} \binom{2r}{2k+1} x^{2r-2k-1} F_k(x) \\ &- \sum_{k=0}^{2r} \binom{2r}{k} (-1)^k x_+^{2r-k} T_k(x), \end{aligned} \tag{72}$$

$$\begin{aligned} x_+^{2r+1} \circledast \tan_-^{-1} x = &-\sum_{k=0}^{r} \binom{2r+1}{2k} x^{2r-2k+1} G_k(-x) - \sum_{k=0}^{r} \binom{2r+1}{2k+1} x^{2r-2k} F_k(-x) \\ &- \sum_{k=0}^{2r+1} \binom{2r+1}{k} x_-^{2r-k+1} T_k(x), \end{aligned} \tag{73}$$

$$\begin{aligned} x_+^{2r} \circledast \tan_-^{-1} x = &-\sum_{k=0}^{r} \binom{2r}{2k} x^{2r-2k} G_k(-x) - \sum_{k=0}^{r-1} \binom{2r}{2k+1} x^{2r-2k-1} F_k(-x) \\ &+ \sum_{k=0}^{2r} \binom{2r}{k} x_-^{2r-k} T_k(x), \end{aligned} \tag{74}$$

for $r = 0, 1, 2, \ldots$.

Proof. We have from equation (2) that

$$x^r \circledast \tan_+^{-1} x = (x_+^r \circledast \tan_+^{-1} x) + (-1)^r (x_-^r \circledast \tan_+^{-1} x), \tag{75}$$

for $r = 0, 1, 2, \ldots$.

Equation (71) follows from equations (19), (58) and (75).

Equation (72) follows from equations (19), (59) and (75).

Equations (73) and (74) follow on replacing x by $-x$ in equations (71) and (72), respectively, on noting that

$$T_k(-x) = (-1)^k T_k(x).$$

□

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